6. Conics

- A parabola is defined as the locus of a point P equidistant from a fixed point (called focus) and a fixed line (called directrix).

- The standard equation of the horizontal parabola is y2=4ax.

- The constant ratio is called the eccentricity and is denoted by *e*. When the eccentricity is unity; *e* = 1, the conic is called a Parabola.

- The line which passes through the focus and perpendicular to the directrix is called axis of the parabola.

- The vertex of a parabola is defined as the intersection point of the parabola and its axis.

- The chord passing through the focus and perpendicular to the axis is called latus rectum.

- Any chord which is perpendicular to the axis of the parabola is called double ordinate.

- The straight line passing through the vertex and perpendicular to the axis of the parabola is called tangent at vertex.

- The end points of the latus rectum are L1a, 2a and L2a,-2a.

- Equation of the Parabola in Non-standard Form

	$(y - k)^2 = 4a\,(x - h)$	$(x - h)^2 = 4b\,(y - k)$
Vertex	(h, k)	(h, k)
Focus	$(a + h, k)$	$(h, b + k)$
Equation of the Directrix	$(x - h) + a = 0$	$(y - k) + b = 0$
Equation of the axis	$y = k$	$x = h$
Tangent at the vertex	$x = h$	$y = k$
Equation of latus rectum	$x - a = h$	$y - k = b$
Length of latus rectum	$\|4a\|$	$\|4b\|$
End points of latus rectum	$L_1\,(a + h,\ 2a + k)$ and $L_2\,(a + h,\ -2a + k)$	$L_1(\,-2b + h,\ b + k)$ and $L_2\,(2b + h,\ b + k)$

- The parametric equation of the standard parabola y2=4ax is x=at2, y=2at.

- The general equation of second degree ax2+2hxy+by2+2gx+2fy+c=0 represents a parabola, if abc+2fgh-af2-bg2-ch2≠0 and h2=ab.

- Focal distance of a point P on parabola is defined as the distance between the point P and its focus S.

- For any point P1x1, y1 outside the parabola, we have y12-4ax1>0

- For any point P2x2, y2 inside the parabola, we have y22-4ax2<0

- An ellipse is the locus of a point which moves such that the ratio of its distance from a fixed point and a fixed line is a constant ratio that is less than one. The fixed point is called the focus; the fixed line is called directrix and the constant ratio is called the eccentricity of the ellipse.

- Standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2(1-e^2)$ and $e<1$ is the eccentricity of the ellipse.

- Equation of an ellipse whose centre is (h, k) and axes are parallel to the x-axis and y-axis is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

- General equation of an ellipse whose focus is (h, k); the equation of the directrix is $ax + by + c = 0$ and the eccentricity, e, is $(x-h)^2+(y-k)^2=e^2\times\frac{(ax+by+c)^2}{a^2+b^2}$.

- The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents an ellipse, if $abc+2fgh-af^2-bg^2-ch^2\neq0$ and $h^2<ab$.

- If the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

    - Point x_1, y_1 lies inside the ellipse, if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1<0$.

    - Point x_1, y_1 lies outside the ellipse, if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0$.

- The parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x=a\cos\theta$, $y=b\sin\theta$, $0\leq\theta<2\pi$. Thus, the coordinates of any point on the ellipse can be taken as $a\cos\theta$, $b\sin\theta$.

- The circle described on the major axis of an ellipse as a diameter is called the auxillary circle of the ellipse.

- A hyperbola is the locus of a point which moves such that the ratio of its distance from a fixed point called the focus and a fixed line called the directrix is a constant which is greater than unity. This constant ratio is called the eccentricity of the hyperbola.

- The standard equation of a hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2=a^2(e^2-1)$ and $e>1$ is the eccentricity of the hyperbola.

- General equation of a hyperbola whose focus is (h, k) and the equation of the corresponding directrix is $ax + by + c = 0$, is $(x-h)^2+(y-k)^2=e^2\times\frac{(ax+by+c)^2}{a^2+b^2}$, where e is the eccentricity of the hyperbola.

- If $abc+2fgh-af^2-bg^2-ch^2\neq0$ and $h^2>ab$, then the general equation of second degree **$ax^2+2hxy+by^2+2gx+2fy+c=0$** represents a hyperbola.

- The parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $x=a\sec\theta$, $y=b\tan\theta$, where $0\leq\theta\leq2\pi$.

- The coordinates of any point on the hyperbola may be taken as $a\sec\theta$, $b\tan\theta$. The angle θ is called the eccentric angle of the point on the hyperbola.

- The point x_1, y_1 lies outside or inside the hyperbola if $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0$ or $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0$.

Equation of Tangent in Different Forms
Let us consider the parabola **$y^2=4ax$**.

- Equation of tangent to the parabola at the point x_1, y_1 is $yy_1 = 2a (x + x_1)$.

- Equation of tangent to the parabola at the point at^2, $2at$ is $ty=x + at^2$.

    The point of intersection of the tangents at the points at_1^2, $2at_1$ and at_2^2, $2at_2$ is at_1t_2, $a(t_1+t_2)$.

- Equation of tangent in terms of the slope and condition of tangency:

  The line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.
  Hence, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$. The point of contact of the tangent is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Some Important Propositions on Parabola

- A tangent at any point P on the parabola bisects the angle between the focal chord through P and forms the perpendicular P on the directrix.

- The portion of a tangent to a parabola cut off between the directrix and the parabola subtends a right angle at the focus.

- Tangents at the extremities of any focal chord intersect at right angles on the directrix.

- Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.

- The area of triangle formed by three points on a parabola is twice the area of the triangle formed by corresponding tangents.

Equation of Normals in Different Forms

Let us consider the parabola $y^2 = 4ax$.

- Equation of normal to the parabola at point x_1, y_1 is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

- Equation of normal to the parabola at point $at^2, 2at$ is $y = -tx + 2at + at^3$.

  Point of intersection of the normals to the parabola $y^2 = 4ax$ at points $at_1^2, 2at_1$ and $at_2^2, 2at_2$ is $2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2(t_1 + t_2)$.

  If a normal at point $at_1^2, 2at_1$ meets the parabola again at $at_2^2, 2at_2$, then $t_2 = -t_1 - \frac{2}{t_1}$.

- Equation of normal to the parabola at point $am^2, -2am$ is $y = mx - 2am - am^3$.

Equation of tangent in different forms

Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- Equation of a tangent to the ellipse at the point x_1, y_1 is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

- Equation of a tangent to the ellipse at the point θ, i.e. $a\cos\theta, b\sin\theta$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

- Coordinates of the point of intersection of the tangents to the ellipse at the points $a\cos\theta, b\sin\theta$ and $a\cos\varphi, b\sin\varphi$ are $\frac{a\cos\frac{\theta+\varphi}{2}}{\cos\frac{\theta-\varphi}{2}}, \frac{b\sin\frac{\theta+\varphi}{2}}{\cos\frac{\theta-\varphi}{2}}$.

- If $c^2 = a^2 m^2 + b^2$, then the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Putting $c = \pm\sqrt{a^2 m^2 + b^2}$ in $y = mx + c$, we get the equation of the tangent to the ellipse in slope form as $y = mx \pm \sqrt{a^2 m^2 + b^2}$.

- The coordinates of the point of contact are $\frac{-a^2 m}{\sqrt{a^2 m^2 + b^2}}, \frac{b^2}{\sqrt{a^2 m^2 + b^2}}$ or $\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \frac{-b^2}{\sqrt{a^2 m^2 + b^2}}$.

- The equation of the director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$.

Equation of the normal in different forms

Let the equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

- Equation of the normal to the ellipse at point x_1, y_1 is $\frac{a^2\,x}{x_1}-\frac{b^2\,y}{y_1}=a^2-b^2$.

- Equation of the normal to the ellipse at point $a\cos\theta, b\sin\theta$ is $ax\sec\theta-by\cosec\theta=a^2-b^2$.

- Equation of the normal to the ellipse in terms of the slope m is $y=mx-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

Equation of the tangent in different forms

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

- Equation of the tangent to the hyperbola at the point $P x_1, y_1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

- Equation of the tangent to the hyperbola at the point $a\sec\theta, b\tan\theta$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

- If $c^2=a^2m^2-b^2$, then the straight line $y=mx+c$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Thus, the equation of the tangent to the hyperbola in slope form is **$y=mx\pm\sqrt{a^2m^2-b^2}$**.

- The coordinates of the point of contact are $\left(-\frac{a^2m}{\sqrt{a^2m^2-b^2}}, -\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$ or $\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

- The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$.

Equation of the normal in different forms

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

- Equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point x_1, y_1 is $\frac{a^2\,x}{x_1}+\frac{b^2\,y}{y_1}=a^2+b^2$.

- Equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $a\sec\theta, b\tan\theta$ is $ax\cos\theta+by\cot\theta=a^2+b^2$.

- Equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, having slope m, is $y=mx\pm\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$.

- Coordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$.

In general, four normals can be drawn from a point in the plane of the hyperbola. These four points are called the **co-normal points.**

Number of Tangents :

1) The equation of a tangent to a parabola $y^2 = 4ax$ with slope m is $y = mx + \frac{a}{m}$.

If this tangent passes through $P(x_1, y_1)$, then

$y_1 = mx_1 + \frac{a}{m} \Rightarrow m^2x_1 - my_1 + a = 0 \quad \ldots 1$

This is a quadratic equation in m. Therefore, at most, two tangents can be drawn to a parabola from a given point in its plane.

2). The equation of a tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with slope *m* is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

If this tangent passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2} \Rightarrow x_1^2 - a^2m^2 - 2x_1y_1m + y_1^2 - b^2 = 0$$

This is a quadratic equation in *m*. Therefore, at most, two tangents can be drawn to the ellipse from a given point in its plane.

3). The equation of a tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with slope *m* is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

If this tangent passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2} \Rightarrow x_1^2 - a^2m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$$

This is a quadratic equation in *m*. Therefore, at most, two tangents can be drawn to the hyperbola from a given point in its plane.**Locus of the point of intersection of perpendicular tangents**

1) The equation of the locus of the point of intersection of mutually perpendicular tangents to the parabola $y^2 = 4ax$ is $x = -a$. It is also the directrix of the parabola.

2) The equation of the locus of the point of intersection of mutually perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x_1^2 + y_1^2 = a^2 + b^2$. It is the circle with the centre at the origin and radius $\sqrt{a^2 + b^2}$ and is the director circle of the given ellipse.

3) The equation of the locus of the point of intersection of mutually perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x_1^2 + y_1^2 = a^2 - b^2$. It is the circle with the centre at the origin and radius $\sqrt{a^2 - b^2}$ and is the director circle of the given hyperbola.
Note:
Since the radius of the director circle is $\sqrt{a^2 - b^2}$, the circle is real whenever $b^2 < a^2$.

If $b^2 = a^2$, the radius becomes zero and it reduces to a point circle at the origin. In this case, the centre of the hyperbola is the only point from which perpendicular tangents can be drawn to the curve.

If $b^2 > a^2$, the radius of the director circle is imaginary and no perpendicular tangents can be drawn to the hyperbola.